Math 60 10.2 Solving Quadratic Equations Using the Quadratic Formula

Objectives 1) Solve quadratic equations using the quadratic formula

2) Use the discriminant $D = b^2 - 4ac$ to determine

- how many solutions the equation has
- whether the solutions are real-rational, real-irrational or complex.

* Note: These problems can (and should) be done without solving the equation!

OPTIONAL: Where does the quadratic formula come from?

To solve $ax^2 + bx + c = 0$ by CTS!

Here are the details:

$$\frac{ax^2}{a} + \frac{bx}{a} = -\frac{c}{a}$$ move constant to RHS, divide by a

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

CTS $\begin{cases} \frac{1}{2} \cdot \frac{b}{a} = \frac{b}{2a} \leftarrow \text{\# for factor} \\ \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} \leftarrow \text{\# to add} \end{cases}$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$ write factor on LHS. find CD on RHS

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$ subtract fractions

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$ square root property

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

* $$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$ add/subtract fractions on RHS.

Quadratic formula to solve $ax^2 + bx + c = 0$

* MUST MEMORIZE!!!

Math 60 10.2

Solve using the quadratic formula

① $12x^2 + 5x - 3 = 0$

step 1: Arrange the equation in standard form, $= 0$.

$ax^2 + bx + c = 0.$

step 2: Identify $a = 12$
$b = 5$
$c = -3$

step 3: Write the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

step 4: Substitute values for a, b, and c.

* CAUTION * Be very careful about negatives!!

$$x = \frac{-5 \pm \sqrt{5^2 - 4(12)(-3)}}{2(12)}$$

Notice: $-4(12)(-3)$ has 2 negatives!

$$x = \frac{-5 \pm \sqrt{25 + 144}}{24}$$

$$x = \frac{-5 \pm \sqrt{169}}{24}$$

simplify using the order of operations: multiply 4ac before subtract.

$$x = \frac{-5 \pm 13}{24}$$

$$x = \frac{-5 + 13}{24}, \frac{-5 - 13}{24}$$

$$x = \frac{8}{24}, \frac{-18}{24}$$

$$\boxed{x = \frac{1}{3}, \frac{-3}{4}}$$

Once you have correctly memorized the formula, the "only" thing that can go wrong is a whole lot of arithmetic!!

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(2) Solve by the quadratic formula

$3p^2 = 6p - 1$

$3p^2 - 6p + 1 = 0$ set $= 0$. arrange in standard form.

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 3$
$b = -6$
$c = 1$

Notice: The variable name goes on the LHS, even when it's not x.

$$p = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(1)}}{2(3)}$$

subst values for a, b, c, esp. using ( ) on $b = -6$.

$$p = \frac{6 \pm \sqrt{36 - 12}}{6}$$

simplify using the order of operations

$$p = \frac{6 \pm \sqrt{24}}{6}$$

$$p = \frac{6 \pm \sqrt{4 \cdot 6}}{6}$$

simplify radical

$$p = \frac{6 \pm 2\sqrt{6}}{6}$$

$$p = \frac{6}{6} \pm \frac{2\sqrt{6}}{6}$$

$$\boxed{p = 1 \pm \frac{\sqrt{6}}{3}}$$

* CAUTION * If you want to write a single fraction

$$\frac{6 + 2\sqrt{6}}{6} = \frac{2(3 + \sqrt{6})}{6}$$

factor

$$= \boxed{\frac{3 + \sqrt{6}}{3}}$$

reduce factor $\frac{2}{6} = \frac{1}{3}$

Math 60 10.2

(3) Solve $9m + \frac{4}{m} = 12$

Technically this is not currently a quadratic equation! Clear fractions by multiplying by LCD = M

$$9m \cdot m + \frac{4}{\cancel{m}} \cdot \cancel{m} = 12 \cdot m$$

$$9m^2 + 4 = 12m$$ set = 0

$$9m^2 - 12m + 4 = $$

$a = 9$
$b = -12$
$c = 4$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(9)(4)}}{2(9)}$$ substitute

$$m = \frac{12 \pm \sqrt{144 - 144}}{18}$$ simplify

$$m = \frac{12 \pm \sqrt{0}}{18}$$

$$m = \frac{12 \pm 0}{18}$$

Note: When we get $\sqrt{0}$, there is only one answer!

$$m = \frac{12}{18} = \boxed{\frac{2}{3}}$$

check for extraneous
$m \neq 0$ b/c it would cause $\div 0$

(4) Solve by the quadratic formula

$$y^2 - 4y + 13 = 0$$

$a = 1$
$b = -4$
$c = 13$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$ substitute

$$y = \frac{4 \pm \sqrt{16 - 52}}{2}$$ simplify

cont ⇒

$$y = \frac{4 \pm \sqrt{-36}}{2}$$

$$y = \frac{4 \pm 6i}{2}$$

$$y = \frac{4}{2} \pm \frac{6}{2}i$$

$$\boxed{y = 2 \pm 3i}$$

The discriminant $D = b^2 - 4ac$ is the number inside the square root of the quadratic formula.

CAUTION Although the logic we'll use involves the square root in the formula, the value of D itself does not have a square root.

Notice in our examples

① $12x^2 + 5x - 3 = 0$
Solutions: $\frac{1}{3}, \frac{-3}{4}$

$D = b^2 - 4ac$
$D = 169$

$D > 0$
2 real solutions
D perfect square
rational
MXL: $\boxed{\text{2 rational solutions}}$

② $3p^2 - 6p + 1 = 0$
Solutions: $1 + \frac{\sqrt{6}}{3}, 1 - \frac{\sqrt{6}}{3}$

$D = 24$

$D > 0$
2 real solutions
D not a perfect sq.
irrational
MXL: $\boxed{\text{2 irrational Solutions}}$

③ $9m^2 - 12m + 4 = 0$
Solutions: $\frac{2}{3}$

$D = 0$

$D = 0$
1 real solution
rational
MXL: $\boxed{\text{one repeated real solution}}$

④ $y^2 - 4y + 13 = 0$
Solutions: $2 + 3i, 2 - 3i$

$D = -36$

$D < 0$
2 complex solutions which are conjugates
MXL: $\boxed{\text{2 complex solutions that are not real}}$

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step 1:

Calculate $D = b^2 - 4ac$

step 2: Ask:

Is D positive? $D > 0$

Is D a perfect square? — step 3 (if necessary)

2 (real) rational solutions

(no $\sqrt{\ }$ remains)

Is D not a perfect square?

$\Downarrow$

2 (real) irrational solutions

($\sqrt{\ }$ remains)

Is D zero? $D = 0$

$\Downarrow$

1 (real,) repeated (rational) solution

(no $\sqrt{\ }$)

Is D negative? $D < 0$

$\Downarrow$

2 complex conjugate solutions (not real)

(doesn't matter if $\sqrt{\ }$ remains, because $i = \sqrt{-1}$ makes it complex)

Calculate the discriminant and determine the number and nature of the solutions, also called roots.

(5) $x^2 - 5x + 2 = 0$

$$\begin{aligned} D &= b^2 - 4ac \\ &= (-5)^2 - 4(1)(2) \\ &= 25 - 8 \\ &= 17 \end{aligned}$$

D is positive, but not a perfect square

$\boxed{\text{2 (real) irrational solutions}}$

(6) $9y^2 + 6y + 1 = 0$

$$\begin{aligned} D = b^2 - 4ac &= 6^2 - 4(9)(1) \\ &= 36 - 36 \\ &= 0 \end{aligned}$$

D is 0 $\Rightarrow$ $\boxed{\text{1 (real rational) repeated solution}}$

Math 60 10.2

(7) $3p^2 - p = -5$

arrange to standard form $= 0$ first!

$3p^2 - p + 5 = 0$

$D = b^2 - 4ac$

$= (-1)^2 - 4(3)(5)$

$= 1 - 60$

$= -59$

D is neg $\Rightarrow$ $\boxed{\text{2 complex (conjugate) solutions that are not real}}$

(8) The revenue R received by a company selling x T-shirts per week is given by $R(x) = -0.005x^2 + 30x$.

a) How many T-shirts must be sold for revenue to be \$25000 per week?

$25000 = -0.005x^2 + 30x$ Subst $R = 25000$

$0.005x^2 - 30x + 25000 = 0$ Set $= 0$

Let's use the discriminant to see if this will work out.

$D = b^2 - 4ac$

$D = (-30)^2 - 4(0.005)(25000)$

$= 900 - 500$

$= 400$ → yes, it will have 2 real, rational solutions.

$x = \dfrac{-b \pm \sqrt{D}}{2a}$ ← we've already done D so we'll re-use that work.

$x = \dfrac{-(-30) \pm \sqrt{400}}{2(.005)}$

$x = \dfrac{30 + 20}{.01}$ $\dfrac{30 - 20}{.01}$

$\boxed{x = 5000, 1000 \text{ T shirts}}$ both sales amounts produce $R = \$25000$

Math 60 10.2

skip b) How many T-shirts must be sold for revenue to be \$45,000 per week

$45000 = -0.005x^2 + 30x$ subst $R = 45000$

$0.005x^2 - 30x + 45000 = 0$ set $= 0$

$$D = b^2 - 4ac$$
$$= (-30)^2 - 4(.005)(45000)$$
$$= 900 - 900 = 0$$

has one solution.

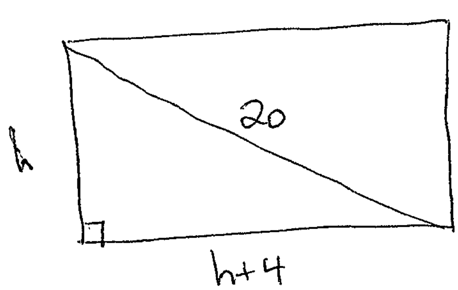
$$x = \frac{-b \pm \sqrt{D}}{2a}$$

$$x = \frac{-(-30) + \sqrt{0}}{2(.005)}$$

$$= \frac{30}{.01}$$

$$= \boxed{3000 \text{ T-shirts}}$$

(9) A designer wants a window so the diagonal is 20 ft, where the window is 4 ft more than the height. What are the dimensions of the window?



Pythagorean Theorem
$a^2 + b^2 = c^2$

Don't forget to square RHS!

$$h^2 + (h+4)^2 = 20^2$$
$$h^2 + h^2 + 8h + 16 = 400$$
$$\frac{2h^2}{2} + \frac{8h}{2} - \frac{384}{2} = \frac{0}{2}$$
$$h^2 + 4h - 192 = 0$$

← We can factor, use CTS, or the quadratic formula.

$$D = b^2 - 4ac$$

Math 60 10.2

$$D = 4^2 - 4(1)(-192)$$
$$= 16 + 786$$
$$= 784$$

$\sqrt{784} = 28$!

$$h = \frac{-b \pm \sqrt{D}}{2(a)}$$
$$= \frac{-4 \pm \sqrt{784}}{2(1)}$$
$$= \frac{-4 \pm 28}{2}$$
$$= \frac{-4}{2} \pm \frac{28}{2}$$
$$= -2 \pm 14$$
$$= -2+14, -2-14$$
$$= 12, -16$$

negative height is not possible!
Reject $h = -16$ as extraneous.

$$\boxed{\begin{array}{l} \text{height} = 12 \text{ ft} \\ \text{width} = h+4 = 16 \text{ ft} \end{array}}$$